From Homology to Derived Functors

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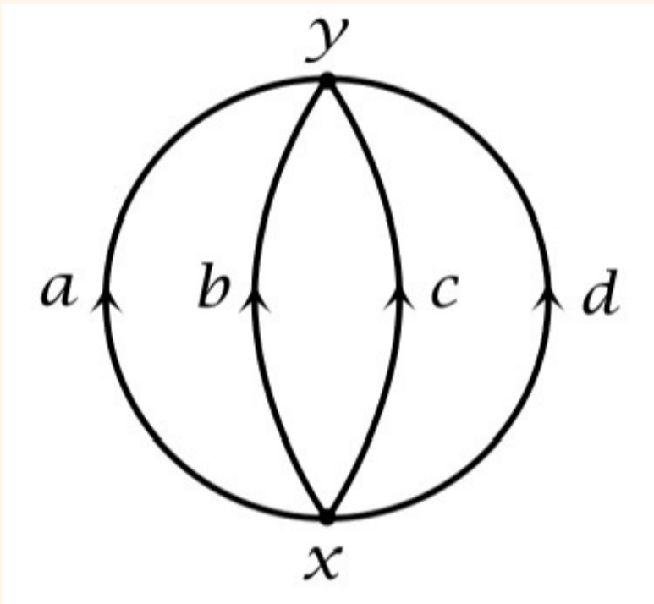
Abstract

This project explores the origins and development of derived functors and their role in modern homological algebra. Using simplicial topology as a motivating example, we demonstrate how the concept of chain complexes arises naturally from geometric intuition. We then generalize this algebraic result to define derived functors. Taking Tor as an example, we discussed how the geometric insights and other properties are reflected via this process.

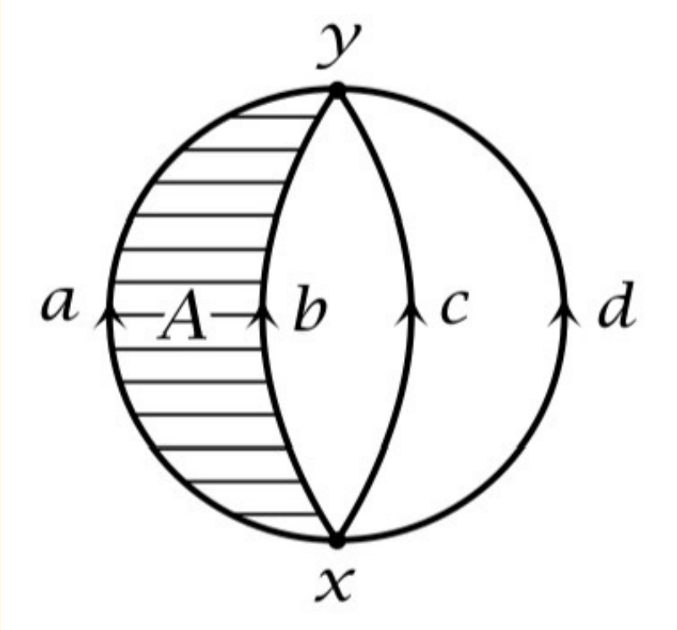
Simplicial Homology

From the study of topology, we know that the fundamental group, or equivalently, the homotopies, reflects the essence of the topological space. Since the fundamental groups are insufficient when it comes to higher dimensions, with much effort, mathematicians developed the idea of homology.

Consider this figure consisting of four oriented edges, the homotopies are different cycles, which form a free abelian group with basis a,b,c,d.



(Hatcher, 2002, pp 99)



(Hatcher, 2002, pp 100)

Suppose the graph is enlarged by attaching a 2-cell within the area. Then, (a-b) will be null-homotopic. This suggests that we may consider the homotopies by quotienting the ideal generated by <a-b>. In such a case, for example, we see that a-c=b-c.

Algebraically, we may interpete this precess by

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

Where the *C*'s are the group of simplicies of corresponding dimensions, $\partial_2(A) = a - b$

Simplicial homology is the homology defined for Δ-complexes.Δ-simplecies is the higher-dimensional analog of triangles. For example, 1-, 2-, 4- simplecies are just line segments, triangles, and tetrahedra, respectively.

Definition.

An n-simplex Δ^n consists of (n+1) vertices. If we remove any one of them, the remaining ones form an (n-1)-simplex, which is called a face. All faces of Δ^n form the boundary $\partial\Delta^n$.

Definition.

A Δ-complex structure on a given space X is a collection of maps: $\sigma_\alpha : \Delta^n \to X$, with certain requirements such that when restricted to a face, the map becomes to some other $\sigma_\beta : \Delta^{n-1} \to X$ in the collection.

Notation. Fix any n, the maps within the structure are denoted by $\Delta^n(X)$. To study the structure algebraically, we denote an n-simplex by $[v_0, v_1, \cdots, v_n]$.

Fact. $\Delta^n(X)$ forms a free abelian group and its elements are in the form of $\sum_\alpha n_\alpha e^n_\alpha$.

Definition.

The boundary homomorphisms are defined by specifying its value on each face:

$$\partial_n(\sigma_\alpha) \sum_i (-1)^i \sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n]$$

Remark. The orientation is vital. The signals reflects the orientations.

Lemma. The composition $\Delta_n(x) \xrightarrow{\partial_n} \Delta_{n-1}(x) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(x)$ is zero.

Definition.

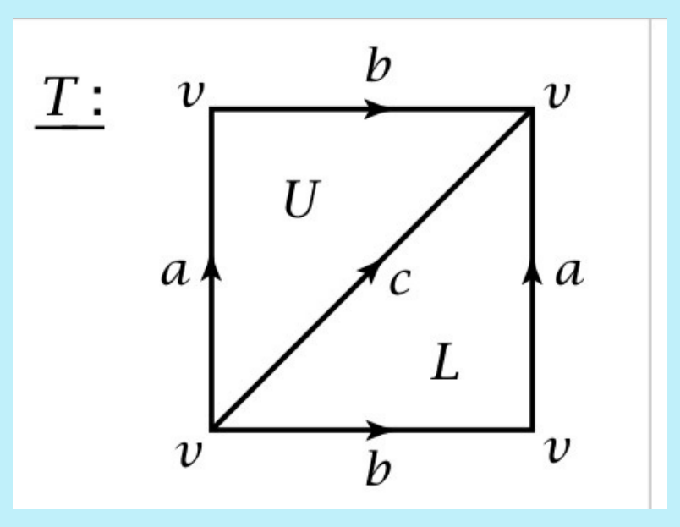
A chain complex is a sequence of objects and maps as follows such that $d^2 = 0$.

$$\cdots \to C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \to \cdots$$

It follows that $\mathbf{Im} d_{n+1} \subset \mathbf{Ker} d_n$, the homology is defined to be $\mathbf{H}_n = \mathbf{Ker} d_n / \mathbf{Im} d_{n+1}$. The chain is called exact if $\mathbf{Im} d_{n+1} = \mathbf{Ker} d_n$.

Example.

For a torus, we may equip it with a Δ-complex structure by the following picture.
It has 1 vertex, 3 edges, and 2 2-simplices.
We compute the homology to be $H_0(T) \cong H_2(T) \cong \mathbb{Z}$, $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $H_n(T) = 0, \forall n \geq 3$.



(Hatcher, 2002, pp 102)

Derived functor

Definition - Comparison Lemma

Given an object *M*, a projective resolution of *M* is a complex *P* together with a map $\epsilon : P_0 \to M$ such that the augmented chain complex is exact.

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M \xrightarrow{0} 0$$

Let $P. \xrightarrow{\epsilon} M$ be a projective resolution of *M*, and we also have a map $f' : M \to N$ Then for every resolution $Q. \xrightarrow{\eta} N$, there exists a chain map lifting f' in the sense that $\eta \circ f_0 = f' \circ \epsilon$. This lifting is unique up to chain homotopic equivalence.

Definition.

Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories and $\mathcal{A}$ has enough projectives (typical examples are modules and abelian groups). Given an object *A*, we may choose any projective resolution $P. \xrightarrow{\epsilon} A$ and define the left derived functors:

$$L_iF(A) = H_i(F(P))$$

Remark. The objects are well defined up to natural isomorphisms, regardless of the choice of *P*. That is, given a second projective resolution $Q. \xrightarrow{\eta} A$, there is a canonical isomorphism:

$$L_iF(A) = H_i(F(P)) \xrightarrow{\cong} H_i(F(Q))$$

Theorem.

Left derived functors form a homological δ-functor. That is, given any short exact sequece $0 \to A' \to A \to A'' \to 0$, there is

$$\cdots \xrightarrow{\partial} L_iF(A') \to L_iF(A) \to L_iF(A'') \xrightarrow{\partial} L_{i-1}F(A') \to \cdots$$

with the $\partial' s$ being natural.

Fact. Taking the opposite category and the dual form of the arguments, one may also obtain injective resolutions and left derived functors.

Example: Tor

Definition.

Let *B* be a left *R*-module, so that $T(-) = - \otimes_R B$ is a right exact functor from **mod**-*R* to **Ab**. We may define the abelian groups

$$\mathrm{Tor}^R_n(A, B) = (L_nT)(A)$$

Lemma. Since $T'(-) = A \otimes_R -$ is also right exact, we may define Tor in a different manner. It turns out that we actually have

$$(L_* A \otimes_R)(B) \cong (L_* \otimes_R B)(A)$$

This explains why in the notation of Tor, we do not specify which of the modules is the functor, and which is the object.

Example. Given an abelian group B, $\mathrm{Tor}^{\mathbb{Z}}_0(\mathbb{Z}/p, B) = B/pB$, $\mathrm{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/p, B) = \{b \in B : pb = 0\}$, and $\mathrm{Tor}^{\mathbb{Z}}_n(\mathbb{Z}/p, B) = 0 \text{ for } n \geq 2.$

Lemma. The following conditions are equivalent for any left module B:

(i) B is flat; (ii) $\mathrm{Tor}^R_n(A, B) = 0, \forall n \neq 0, \forall A$; (iii) $\mathrm{Tor}^R_1(A, B) = 0, \forall A$.

Discussion

Homology provides a powerful bridge between geometry and algebra. Starting from the intuitive idea of detecting holes in a space, simplicial homology translates geometric features into algebraic data through chain complexes and boundary maps. Computable is the key here. It provides us with a new perspective to handle topological problems, especially when they become less intuitive.

Derived functors, Tor being the example discussed, emerge when extending these ideas into more general algebraic settings. Tor measures how this process fails when modules are not "flat", and thus cannot be easily merged into one abelian group. In algebraic geometry, it reflects how subspaces intersect, capturing multiplicities and tangencies.

This tool appears as addition and multiplication, being ubiquitous and indispensable in various fields.

References:

Hatcher, A. (2002). Algebraic topology. Cambridge University Press.

Weibel, C. A. (1994). An introduction to homological algebra. Cambridge University Press.